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ON THE THEORY OF STABILITY OF PROCESSES OVER A SPECIFIED TIME INTERVAL

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The present paper supplements and formulates in a more rigorous form, the statement of the problem on stability of processes over a specified time interval, which was given in /1, 2/. The refinement concerns the case in which the specified time interval is finite, and we find that an imposition of stronger constraints on the region of limiting deviations becomes necessary. As far as the character of the constraints imposed on the perturbations of the parameters of the process is concerned, the proposed formulation and the initial formulation are both related to /3/. We use the fact that a linear differential system can be transformed into a diagonal one, as the basis for establishing the necessary and sufficient conditions of stability of a linear process, and for obtaining certain conditions of stability and instability of a nonlinear process in the linear approximation. We show how transformation of a linear system to a "nearly" diagonal system can be utilized for the same purpose.

1. Choice of the region of limiting deviations. We introduce the region of limiting deviations using the class K_Δ^ω of $(n \times n)$-matrices $G(t) = (G_1 G_2 \ldots G_n)$ over the field of complex numbers, satisfying the following conditions on the interval $\Delta = [t_0, T)$, where T is a number greater than t_0, or ∞ : $\det G(t) \neq 0$ and the Hermitian norm of the columns $G_j(t)$ $(j = 1, 2, \ldots, n)$ coincides with a positive function $\omega(t)$, i.e. $\| G_j(t) \| = \sqrt{(G_j, G_j)} = \omega(t)$.

The region of limiting deviations is defined as follows:

$$(G^{-1}(t)\,x,\ G^{-1}(t)\,x) \leqslant \rho^2, \quad G(t) \in K_\Delta^\omega \tag{1.1}$$

Here x is a column matrix of the deviations $x_1, x_2, \ldots, x_n$ from the nominal values of the parameters characterizing the process under investigation, and ρ is a positive number. The left-hand side of the relation (1.1) contains the Hermitian form of the coordinates $x_1, x_2, \ldots, x_n$, which assumes real nonnegative values only for any value of x. Geometrically, the relation (1.1) represents, for each fixed t in the $x_1, x_2, \ldots, x_n$-coordinate space, an n-dimensional ellipsoid bounded by the surface

$$(G^{-1}(t)\,x, \quad G^{-1}(t)\,x) = \rho^2 \tag{1.2}$$

possessing the following properties:

Each of the $2n$ rays $x = \pm G_\sigma(t)\,s$ $(\sigma = 1, 2, \ldots, n;\ s > 0)$, where G_σ $(\sigma = 1, 2, \ldots, n)$ are columns of the matrix G, intersects the surface (1.2) once when the parameter $s = \rho$. The points of intersection of these rays with the surface (1.2) are situated at the distance $\rho_\omega = \omega\rho$ from the coordinate origin $(x = 0)$.

Plane $x = G_i s_i + G_j s_j$ $(i \neq j)$ is generated by any pair of columns of the matrix G and it intersects the surface (1.2) along an ellipse defined by the equations

$$x = G_i s_i + G_j s_j, \qquad s_i^2 + s_j^2 = \rho^2 \tag{1.3}$$

The rays $G_i s_i$ and $G_j s_j$ are symmetric with respect to the principal axes of the ellipse (1.3) and directed along the diagonals of a rectangle the sides of which touch the ellipse (1.3) at its apexes $\pm{}^1/_2\sqrt{2}\,(G_i \pm G_j)$. The semiaxes of the ellipsoid (1.2) are $a_i = \sqrt{\mu_i}\rho$ $(i = 1, 2, \ldots, n)$, where μ_i are the eigenvalues of the Hermitian matrix G^*G and $0 < a_i < \sqrt{2}\omega\rho$. Equation (1.2) defines, in the $(n+1)$-dimensional $x_1, x_2, \ldots, x_n$-coordinate space and time t, a tube (ρ_ω tube) each intersection of which with a hyperplane $t = t^*$ represents an ellipsoid with the properties indicated above. The orientatiion of the principal axes of this ellipsoid may vary arbitrarily with time, and the ellipsoid itself may become deformed (i.e. the dimensions of its semiaxes may change); at the same time the distance between the coordinate origin and the points of intersection of all rays $\pm G_\sigma(t)\,s$ with the surface of the ellipsoid assume strictly defined values, in particular when $\omega = \mathrm{const}$, the distance remains unchanged.

2. Definitions. The following definitions were given in /1, 2/ for the region of limiting deviations of the form (1.1):

Definition 1. If in a given class K_Δ^ω a matrix $G(t)$ exists such that for sufficiently small $\rho > 0$ any perturbation of the process, the initial value $x_0 = x(t_0)$ of which satisfies the condition

$$(G^{-1}(t_0)\,x_0,\ G^{-1}(t_0)\,x_0) \leqslant \rho^2$$

on the interval $\Delta = [t_0, T)$, satisfies the condition

$$(G^{-1}(t_0)\,x_0, \quad G^{-1}(t_0)\,x_0) \leqslant \rho^2$$

then the unperturbed process is stable on the interval $[t_0, T)$, otherwise it is unstable.

If we now consider the conditions of stability and instability of the solution $x \equiv 0$ on the interval $[t_0, \infty)$ in the linear approximation which follow from the formulation given above, as applied to the dynamic systems described by an equation of the form

$$dx/dt = U(t)\,x + h(t, x) \tag{2.1}$$

where x is a column matrix of the perturbations, U is a square matrix of order n continuous on $[t_0, T)$ and h is a column matrix with the property

$$\frac{h(t, x)}{\|x\|} \underset{t}{\rightrightarrows} 0 \quad \text{for} \quad x \to 0 \tag{2.2}$$

we find that in many aspects these conditions are analogous to the corresponding results of the Liapunov theory. In particular, when $U = \text{const}$ and the real parts of all eigenvalues of the matrix U are negative, then the trivial solution of Eq. (2.1) is stable; if on the other hand at least one of the eigenvalues of U has a positive real part, then the solution is unstable. In general, the stability of the solution $x \equiv 0$ (on $[t_0, \infty)$) in the sense of Definition 1 implies the Liapunov stability, the converse however is not always true.

Direct application of the concept of stability in the form given by Definition 1 leads, in the case of a finite interval $(T < \infty)$ to an unacceptable conclusion. Thus, when $U = \text{const}$, the presence of even a single eigenvalue of U with a negative real part can represent a sufficient condition of stability of the solution $x \equiv 0$ of (2.1) on a finite interval, in the sense of Definition 1, irrespective of what the remaining eigenvalues are. This implies that the constraints imposed in Definition 1 on the region of limiting deviations which are natural in the case of an infinite time interval, must be supplemented in the case of a finite time interval. In this connection we propose the following modification of Definition 1, which is preferable in the case of a finite time interval.

Definition 2. If in a given class K_Δ^ω a matrix $G(t)$ exists coinciding at the instant $t = t_0$ with a given constant matrix G_0 of class $K_\Delta{}^{\omega(t_0)}$ and such that for a sufficiently small $\rho > 0$ an arbitrary perturbation $x(t)$ of the process, the initial value $x_0 = x(t_0)$ of which satisfies the condition

$$(G_0^{-1}x_0,\ G_0^{-1}x_0) \leqslant \rho^2 \tag{2.3}$$

on the interval $\Delta = [t_0, T)$, satisfies the condition

$$(G^{-1}(t)x,\ G^{-1}(t)x) \leqslant \rho^2$$

then the uperturbed process is stable on the interval $[t_0, T)$, otherwise it is unstable.

Below we give some of the conditions, in the sense of Definition 2, for the processes described by equations of the form (2.1).

3. Linear systems. We have

$$dx/dt = U(t)x \tag{3.1}$$

where U is a square matrix of order n continuous on $[t_0, T)$. Change of the variables

$$x = K(t)\,y, \quad K(t) = X(t)\,G_0 Z(t) \quad \left(\frac{dX}{dt} = UX,\ X(t_0) = E\right) \tag{3.2}$$

where X is a solution of the matrix equation given in brackets, E is the unit matrix $G_0 = (g_1^\circ, g_2^\circ \ldots g_n^\circ)$ is a given constant matrix of class $K_\Delta{}^{\omega(t_0)}$ and Z is a diagonal matrix continuously differentiable and nondegenerate on $[t_0, T)$, reduces (3.1) to the diagonal form (see /2/)

$$dy/dt = \Lambda(t)y, \quad \Lambda(t) = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{3.3}$$

It can be stipulated here that all columns K_σ ($\sigma = 1, 2, \ldots, n$) of the matrix K have the same single Hermitian norm $\alpha(t)$, i.e. that

$$\| K_\sigma(t) \| = \alpha(t) > 0, \quad \sigma = 1, 2, \ldots, n$$

Then Z and Λ are given by the following formulas:

$$Z = \alpha(t)\,\mathrm{diag}\left(\frac{e^{i\theta_1}}{\| Xg_1^\circ \|}, \ldots, \frac{e^{i\theta_n}}{\| Xg_n^\circ \|}\right)$$

$$\mathrm{Re}\,\Lambda = \frac{d}{dt}\ln \mathrm{diag}\left(\frac{\| Xg_1^\circ \|}{\alpha}, \ldots, \frac{\| Xg_n^\circ \|}{\alpha}\right)$$

$$\mathrm{Im}\,\Lambda = -\,\mathrm{diag}\left(\frac{d\theta_1}{dt}, \ldots, \frac{d\theta_n}{dt}\right)$$

where θ_j ($j = 1, 2, \ldots, n$) are arbitrary continuously differentiable real scalar functions.

In what follows we shall assume $\theta_j \equiv 0$ ($j = 1, 2, \ldots, n$). It can easily be seen that in this case $K(t_0) = G_0$ when $\alpha(t_0) = \omega(t_0)$. When the matrix $K(t)$ is chosen in this manner, the bundle of solutions of (3.1) satisfying the condition (2.3) is represented by the relation

$$(H^{-1}(t)x,\ H^{-1}(t)x) \leqslant \rho^2, \quad t \in [t_0, T) \tag{3.4}$$

where $H = (H_1 H_2 \ldots H_n)$ is a square matrix of order n, defined by the expression

$$HH^* = K(t)\exp\left(\int_{t_0}^{t} 2\mathrm{Re}\,\Lambda\,dt\right)K^*(t) \tag{3.5}$$

under the condition that all columns H_σ ($\sigma = 1, 2, \ldots, n$) have the same single norm $\omega_0(t)$ for each t. The matrix $G(t) = (\omega(t)/\omega_0(t))H$ obviously belongs to the class K_Δ^ω. The corresponding ρ_ω-tube

$$(G^{-1}(t)x,\ G^{-1}(t)x) = \rho^2 \tag{3.6}$$

is characterized by the fact that each of its cross sections $t = t^*$ is an ellipsoid similar to the ellipsoid

$$(H^{-1}(t)x,\ H^{-1}(t)x) = \rho^2 \tag{3.7}$$

and the directions of the principal axes of the ellipsoids (3.6) and (3.7) in the $x_1, x_2, \ldots, x_n$-coordinate space coincide. We shall call this ρ_ω-tube an "associated" tube in the sense defined above.

When $t = t_0$ we have $H(t_0) = K(t_0) = G_0$, and at this instant the associated ρ_ω-tube coincides with the envelope of the bundle of solutions (3.4). The surfaces (3.6) and (3.7) coincide also at the values of t for which $\omega_0(t) = \omega(t)$. When $\omega_0(t) < \omega(t)$, the ellipsoid (3.7) is found to be wholly contained within the ellipsoid (3.6), conversely when $\omega_0(t) > \omega(t)$, then the ellipsoid (3.6) is contained within the ellipsoid (3.7). It follows that when $\omega_0(t) \leqslant \omega(t)$ all solutions of the linear system belonging to the bundle (3.4) fall within the confinements of the associated ρ_ω-tube. If on the other hand $\omega_0(t) > \omega(t)$, then some of the solutions belonging to the bundle (3.4) will invariably pass outside the limits of the associated ρ_ω-tube. It can be shown that when $\omega_0(t) > \omega(t)$ and we replace the associated ρ_ω-tube by any other ρ_ω-tube, some of the solutions belonging to the bundle (3.4), i.e. the solutions of the linear system which were situated inside or on the surface of the ellipsoid (2.3) at the initial instant of time t_0, will always be found outside this tube. From all this follows

Theorem 3.1. The necessary and sufficient conditions of stability of the trivial solution of the linear system (3.1) is, that

$$\omega_0(t) \leqslant \omega(t), \quad \forall\, t \in [t_0, T)$$

The Hermitian norm of the columns of the matrix H is given by the relation

$$\omega_0^2(t) \equiv \| H_s(t) \|^2 = \frac{1}{n} \sum_{\sigma=1}^{n} \exp[2\mu_\sigma(t)(t-t_0)]\, \alpha^2(t), \quad s = 1, 2, \ldots, n$$

$$\mu_\sigma(t) = \frac{1}{t-t_0} \int_{t_0}^{t} \operatorname{Re} \lambda_\sigma(t)\, dt$$

Taking this into account, we can formulate a number of simpler conditions of stability and instability of an unperturbed process.

Corollaries. 1°. Let

$$\alpha(t) \leqslant \omega(t), \quad \mu(t) \leqslant 0 \quad (\mu(t) = \max_\sigma \mu_\sigma(t))$$

on the interval $[t_0, T)$. Then the unperturbed process (solution of (3.1)) is stable on $[t_0, T)$.

2°. If at any point $\bar{t} \in [t_0, T)$

$$\alpha(\bar{t}) \geqslant \omega(\bar{t}), \quad \frac{1}{n} \sum_{\sigma=1}^{n} \exp[2\mu_\sigma(\bar{t})(\bar{t}-t_0)] > 1$$

then the unperturbed process (solution of (3.1)) is unstable on $[t_0, T)$.

We shall call the unperturbed process asymptotically stable on $[a, \infty)$ if it is stable on $[a, \infty)$ (in the sense of Definition 2) and if for any $t_0 \in [a, \infty)$ there exists $\rho = \rho(t_0) > 0$ such that all perturbations $x(t)$ of the process satisfying the condition (2.3) have the property $\lim \| x(t) \| = 0$ as $t \to \infty$.

3°. Let

$$\alpha(t) \leqslant \omega(t), \quad \mu(t) < -b, \quad b = \text{const} > 0$$

on the interval $[t_0, \infty)$. Then the unperturbed process (solution of (3.1)) is asymptotically stable on $[t_0, \infty)$.

4°. Let

$$\omega(t) / \alpha(t) \leqslant N, \quad N = \text{const} \geqslant 0$$

on the interval $[t_0, \infty)$ and, beginning from some $t^* > t_0$, let

$$\mu(t) \geqslant b, \quad b = \text{const} > 0$$

Then the unperturbed process (solution of (3.1)) is unstable on $[t_0, \infty)$.

Example. Consider a linear system with constant coefficients ($U = \text{const}$). Let for simplicity U be a matrix of simple structure, with eigenvalues $\nu_1, \nu_2, \ldots, \nu_n$ (not necessarily all different) and the corresponding eigenvectors $\gamma_1, \gamma_2, \ldots, \gamma_n$. In this case the fundamental matrix of the linear equation (3.1) has the form

$$X = \sum_{j=1}^{n} \exp\,[\nu_j (t - t_0)]\, P_j$$

where P_j is the matrix of the orthogonal projection of the n-dimensional space R onto the subspace R_j generated by the eigenvector γ_j. In accordance with this we have

$$\mu_\sigma (t) = \frac{1}{t - t_0} \ln \left\{ \left\| \sum_{j=1}^{n} \exp\,[\nu_j (t - t_0)]\, P_j g_\sigma^\circ \right\| \Big/ \alpha (t) \right\}$$

It is clear from the last expression that the stability or instability of the process depends on ν_i as well as on the prescribed G_0 and $\omega (t)$ (the function $\omega (t)$ restricts the choice of the function $\alpha (t)$). Here we shall only consider the simplest case in which $G_0 = (\gamma_1 \gamma_2 \cdots \gamma_n)$. We also have

$$\mu_\sigma (t) = \operatorname{Re} \nu_\sigma + \frac{1}{t - t_0} \ln \frac{\omega (t_0)}{\alpha (t)}, \quad \sigma = 1, 2, \ldots, n$$

Assuming that $\omega (t)$ is a differentiable function, we set $\alpha (t) \equiv \omega (t)$. Then the unperturbed process will be stable on $[t_0, T)$ if

$$\max_\sigma \operatorname{Re} \nu_\sigma + \frac{1}{t - t_0} \ln \frac{\omega (t_0)}{\omega (t)} \leqslant 0$$

asymptotically stable on $[t_0, \infty)$ if

$$\max_\sigma \operatorname{Re} \nu_\sigma + \frac{1}{t - t_0} \ln \frac{\omega (t_0)}{\omega (t)} < - b$$

and unstable on $[t_0, T)$ if

$$\frac{1}{n} \sum_{\sigma=1}^{n} \exp\,[2 (t - t_0) \operatorname{Re} \nu_\sigma] > \frac{\omega^2 (t)}{\omega^2 (t_0)}$$

4. Nonlinear systems. Stability on a finite interval in the linear approximation. We consider a nonlinear process represented by a trivial solution $x \equiv 0$ of (2.1). We can assume without loss of generality that in the case of $T < \infty$ the matrix K of the transformation of the linear approximation (i.e. of (3.1)) is nondegenerate and differentiable, by virtue of (3.3), on a closed interval $[t_0, T]$.

Remembering that H is, as before, a matrix appearing in the relation (3.4), we introduce the function

$$V (t, x) = \frac{\omega_0^2 (t)}{\omega^2 (t)} (H^{-1} (t)\, x,\ H^{-1} (t)\, x)$$

The equation $V (t, x) = \rho^2$ represents the associated tube, since $H\, \omega / \omega_0 \in K_\Delta^\omega$.

Taking into account (3.5) and carrying out the change of variables defined by (3.2), we obtain

$$V (t, x) = \frac{\omega_0^2 (t)}{\omega^2 (t)} \left(\exp \left[- \int_{t_0}^{t} 2 \operatorname{Re} \Lambda \, dt \right] y,\ y \right) \tag{4.1}$$

From (2.1) we find

$$y = I (t) \left(\int_{t_0}^{t} \frac{Mh}{I (t')} dt' + y (t_0) \right), \quad M = K^{-1}, \quad I (t) = \exp \int_{t_0}^{t} \Lambda \, dt \tag{4.2}$$

Substituting (4.2) into (4.1) yields

$$V (t, x) = \frac{\omega_0^2 (t)}{\omega^2 (t)} V (t_0, x_0) [1 + \psi (t, y)] \tag{4.3}$$

where

$$\psi(t, y) = 2\,\mathrm{Re}\frac{y^*(t_0)}{\|y(t_0)\|}\int_{t_0}^{t}\frac{Mh}{I(t')\|y(t_0)\|}\,dt'$$

By virtue of the condition (2.2) and the nondegeneracy of the matrix K we have, on the closed interval $[t_0, T]$

$$\psi(t, y) \underset{t}{\rightrightarrows} 0 \quad \text{when} \quad y \to 0 \tag{4.4}$$

The relations (4.3) and (4.4) enable us to formulate the following theorems:

Theorem 4.1. If $\omega_0(t) < \omega(t)$ with $\forall t \in [t_0, T]$ $(T < \infty)$, then the unperturbed process (trivial solution of (2.1)) is stable on the finite interval $[t_0, T)$.

Theorem 4.2. If $\omega_0(\bar{t}) > \omega(\bar{t})$ at any point $\bar{t} \in [t_0, T)$, then the unperturbed process (trivial solution of (2.1)) is not stable on the interval $[t_0, T)$.

5. The conditions of stability given above are based on the feasibility of transforming the linear system (3.1) to its diagonal form (3.3). However, the matrix of such transformation contains the basic matrix X (see (3.2)) as a multiplier, and an exact expression cannot always be obtained for the latter matrix in a finite form. In this connection it is advisable to construct the conditions of stability using a transformation converting the linear system to a system nearly diagonal. Methods of constructing matrices of such transformations are well known.

Let us introduce the matrix

$$G(t) = K(t)\, I(t)\, M_0\, G_0 I^{-1}(t)\, \Omega \qquad (M_0 = M(t_0))$$

Here K is a square matrix of order n, nondegenerate and differentiable on $[t_0, T]$, Λ is a diagonal matrix with diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_n$, the above matrices connected with each other and with another matrix N by the relation $dK/dt = UK - K\Lambda + N$; Ω is a normalizing diagonal matrix ensuring that the conditions $\|G_j(t)\| = \alpha(t) > 0$, $j = 1, 2, \ldots, n$ (G_j denote the columns of the matrix G) hold, and $\Omega(t_0)$ is a unit matrix. We then have

Theorem 5.1. Let

$$\alpha(t) \leqslant \omega(t), \quad \frac{1}{t - t_0}\int_{t_0}^{t'}[\mu_0(t') + \nu_{\max}(t')]\,dt' \leqslant -b$$

hold on the interval $[t_0, T < \infty)$. Here $\mu_0(t)$ is the real part of the eigenvalue of the diagonal matrix $\Lambda - \Omega^{-1}\, d\Omega/dt$, at which it assumes its maximum value and $\nu_{\max}(t)$ is the largest eigenvalue of the Hermitian matrix

$$P = -{}^1/_2\,(G^{-1}NMG + G^* M^* N^* G^{*^{-1}})$$

Then the unperturbed process (trival solution of (2.1)) will be stable on the interval $[t_0, T)$. The proof is analogous to that of Theorem 6.3 of /2/.

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